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HIGHER-ORDER DEGENERATE q-BERNOULLI POLYNOMIALS

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Abstract. In this paper, we study the higher-order degenerate q-Bernoulli polynomials which are derived from the p-adic q-integrals on $\mathbb{Z}_p$. In addition, we give some identities and properties for these polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. As is well known, Carlitz's degenerate Bernoulli polynomials are defined by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_n(x \mid \lambda)\frac{t^n}{n!}, \quad (\text{see } [4, 5]). \tag{1.1}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0 \mid \lambda)$ are called the degenerate Benroulli numbers. Note that $\lim_{\lambda\to 0}\beta_n(x \mid \lambda) = B_n(x)$, $(n \geq 0)$, where $B_n(x)$ are called ordinary Bernoulli polynomials.

For $r \in \mathbb{N}$, the higher-order degenerate Bernoulli numbers are also given by the generating function

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_n^{(r)}(x \mid \lambda)\frac{t^n}{n!}. \tag{1.2}$$

When $x = 0$, $\beta_n^{(r)}(0 \mid \lambda) = \beta_n^{(r)}(\lambda)$ are called the degenerate Bernoulli numbers of order r(see [5]). Note that $\lim_{\lambda\to 0}\beta_n^{(r)}(x \mid \lambda) = B_n^{(r)}(x)$, $(n \geq 0)$, where $B_n^{(r)}(x)$ are the higher-order Bernoulli polynomials given by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see } [1\text{–}13]). \tag{1.3}$$

Let $q \in \mathbb{C}_p$ be an indeterminate with $|1-q|_p < p^{-\frac{1}{p-1}}$. Then Carlitz's q-Benroulli numbers are defined as

$$q(q\beta_q+1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases} \quad \beta_{0,q} = 0,, \tag{1.4}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$ (see [9, 10]).

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The q-Bernoulli polynomials are also defined by Carlitz as follows:

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l}, \quad (\text{see } [4, 6, 9]), \tag{1.5}$$

where $[x]_q = \frac{1-q^x}{1-q}$.

Let f be a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim and given by

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x) \, d\mu_q(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x) \mu_q\left(x + p^N \mathbb{Z}_p\right) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see } [9]). \end{aligned} \tag{1.6}$$

From (1.6), we note that

$$q \int_{\mathbb{Z}_p} f(x+1) \, d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) \, d\mu_q(x) + (q-1) f(0) + \frac{q-1}{\log q} f'(0). \tag{1.7}$$

By (1.7), we get

$$\begin{aligned} q \int_{\mathbb{Z}_p} [x+1]_q^n \, d\mu_q(x) - \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x) &= (q-1) 0^n + n 0^{n-1} \\ &= \begin{cases} q-1 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ 0 & \text{if } n>1. \end{cases} \end{aligned} \tag{1.8}$$

It is easy to show that

$$\begin{aligned} [x+1]_q^n &= \left(1 + q [x]_q\right)^n \\ &= \sum_{l=0}^{n} \binom{n}{l} q^l [x]_q^l \end{aligned} \tag{1.9}$$

From (1.8) and (1.9), we have

$$q \sum_{l=0}^{n} \binom{n}{l} q^l \int_{\mathbb{Z}_p} [x]_q^l \, d\mu_q(x) - \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x) = \begin{cases} q-1 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ 0 & \text{if } n>1. \end{cases} \tag{1.10}$$

In [9], Kim proved that Carlitz's q-Bernoulli numbers can be represented by the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x) = \beta_{n,q}, \quad (n \geq 0). \tag{1.11}$$

By (1.10) and (1.11), we get

$$q \sum_{l=0}^{n} \binom{n}{l} q^l \beta_{l,q} - \beta_{n,q} = \begin{cases} q-1 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ 0 & \text{if } n>1. \end{cases} \tag{1.12}$$

The Carlitz's q-Bernoulli polynomials are also written by the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_q(y)&=\sum_{l=0}^{n}\binom{n}{l}q^{lx}\int_{\mathbb{Z}_p}[y]_q^l\,d\mu_q(y)\,[x]_q^{n-l}\\&=\sum_{l=0}^{n}\binom{n}{l}q^{lx}\beta_{l,q}\,[x]_q^{n-l},\quad(n\geq 0).\end{aligned}\tag{1.13}$$

The generating function of Carlitz's q-Bernoulli polynomials is given by

$$\int_{\mathbb{Z}_p}e^{[x+y]_q t}d\mu_q(y)=\sum_{n=0}^{\infty}\beta_{n,q}(x)\frac{t^n}{n!}.\tag{1.14}$$

Note that $e^t=\lim_{\lambda\to 0}(1+\lambda t)^{\frac{1}{\lambda}}$.

Recently, the degenerate q-Bernoulli polynomials are introduced by Kim and given by

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}[x+y]_q}\,d\mu_q(y)=\sum_{n=0}^{\infty}\beta_{n,q}(x\mid\lambda)\frac{t^n}{n!},\quad(\text{see [10]}).\tag{1.15}$$

When $x=0$, $\beta_{n,q}(0\mid\lambda)$ are called the degenerate q-Bernoulli numbers. It is not difficult to show that $\lim_{\lambda\to 0}\beta_{n,q}(x\mid\lambda)=\beta_{n,q}(x)$.

In this paper, we study the degenerate higher-order q-Bernoulli polynomials and numbers which are derived from the p-adic q-integrals on $\mathbb{Z}_p$ and investigate some properties of these polynomials and numbers.

Recently, the q-Bernoulli and degenerate Bernoulli polynomials have been studied by many researchers. (see [1–13]).

2. Higher-order degenerate q-Bernoulli polynomials

Let h,k be positive integers and let us define the degenerate higher-order q-Bernoulli polynomials as follows:

$$\begin{aligned}&\sum_{n=0}^{\infty}\beta_{n,q}^{(h,k)}(x\mid\lambda)\frac{t^n}{n!}\\&=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}(1+\lambda t)^{\frac{[x_1+\cdots+x_k+x]_q}{\lambda}}q^{x_1(h-1)+\cdots+x_k(h-k)}d\mu_q(x_1)\cdots d\mu_q(x_k).\end{aligned}\tag{2.1}$$

Now, we note that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x_1+\cdots+x_k+x]_q}{\lambda}}q^{x_1(h-1)+x_2(h-2)+\cdots+x_k(h-k)}d\mu_q(x_1)\cdots d\mu_q(x_k)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p^k}\binom{\frac{[x_1+\cdots+x_k+x]_q}{\lambda}}{n}q^{x_1(h-1)+x_2(h-2)+\cdots+x_k(h-k)}d\mu_q(x_1)\cdots d\mu_q(x_k)\,\lambda^n t^n\\&=\sum_{n=0}^{\infty}\lambda^n\int_{\mathbb{Z}_p^k}\left(\frac{[x_1+\cdots+x_k+x]}{\lambda}\right)_n q^{x_1(h-1)+\cdots+x_k(h-k)}d\mu_q(x_1)\cdots d\mu_q(x_k)\frac{t^n}{n!},\end{aligned}\tag{2.2}$$

where

$$\int_{\mathbb{Z}_p^k} f(x_1,\ldots,x_k)\,d\mu_q(x_1)\cdots d\mu_q(x_k)$$
$$=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}} f(x_1,\ldots,x_k)\,d\mu_q(x_1)\cdots d\mu_q(x_k),$$

and

$$\begin{aligned}(2.3)\qquad \left(\frac{[x]_q}{\lambda}\right)_n &= \frac{[x]_q}{\lambda}\left(\frac{[x]_q}{\lambda}-1\right)\cdots\left(\frac{[x]_q}{\lambda}-n+1\right)\\ &= \lambda^{-n}[x]_q\left([x]_q-\lambda\right)\left([x]_q-2\lambda\right)\cdots\left([x]_q-(n-1)\lambda\right)\\ &= \lambda^{-n}\left([x]_q \mid \lambda\right)_n.\end{aligned}$$

Therefore, (2.1), (2.2) and (2.3), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-times}\left([x_1+\cdots+x_k+x]_q\Big|\lambda\right)_n$$
$$\times q^{x_1(h-1)+\cdots+x_k(h-k)}d\mu_q(x_1)\cdots d\mu_q(x_k)=\beta_{n,q}^{(h,k)}(x\mid\lambda).$$

We observe that

$$\begin{aligned}(2.4)\qquad \left([x_1+\cdots+x_k+x]_q\Big|\lambda\right)_n &= \lambda^n\left(\frac{[x_1+\cdots+x_k+x]_q}{\lambda}\right)_n\\ &= \lambda^n\sum_{l=0}^{n}S_1(n,l)\left(\frac{[x_1+\cdots+x_k+x]_q}{\lambda}\right)^l\\ &= \sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}[x_1+\cdots+x_k+x]_q^l,\end{aligned}$$

where $S_1(n,l)$ is the Stirling number of the first kind.

From Theorem 1 and (2.4), we note that

(2.5)

$$\begin{aligned}\beta_{n,q}^{(h,k)}(x\mid\lambda) &= \sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}\\ &\quad\times\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}[x_1+\cdots+x_k+x]_q^l\,q^{x_1(h-1)+\cdots+x_k(h-k)}d\mu_q(x_1)\cdots d\mu_q(x_k)\\ &= \sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}\frac{1}{(1-q)^l}\sum_{m=0}^{l}\binom{l}{m}q^{mx}(-1)^m\\ &\quad\times\frac{(m+h)(m+h-1)\cdots(m+h-k+1)}{[m+h]_q[m+h-1]_q\cdots[m+h-k+1]_q}\end{aligned}$$

$$= \sum_{l=0}^{n} \sum_{m=0}^{l} \frac{1}{(1-q)^l} S_1(n,l)\lambda^{n-l} \binom{l}{m} q^{mx} (-1)^m$$
$$\times \frac{(m+h)(m+h-1)\cdots(m+h-k+1)}{[m+h]_q [m+h-1]_q \cdots [m+h-k+1]_q}$$
$$= \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} (-1)^m (1-q)^{k-l} \lambda^{n-l} S_1(n,l) \frac{(m+h)_k}{(q^{m+h}:q^{-1})_k} q^{mx},$$

where $(x:q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})(n \geq 1)$, $(x:q)_0 = 1$.

Therefore, by (2.5), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\beta_{n,q}^{(h,k)}(x \mid \lambda) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} (-1)^m (1-q)^{k-l} \lambda^{n-l} S_1(n,l) \frac{(m+h)_k}{(q^{m+h}:q^{-1})_k} q^{mx}.$$

Note that

$$\lim_{\lambda \to 0} \beta_{n,q}^{(h,k)}(x \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m (1-q)^{k-n} \frac{(m+h)_k}{(q^{m+h}:q^{-1})_k} q^{mx}$$
$$= \beta_{n,q}^{(h,k)}(x), \quad (\text{see } [9, 10]).$$

In [4], L. Carlitz introduced the higher-order q-Bernoulli polynomials which are given by

$$\beta_{m,q}^{(h,k)}(x) = \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{jx} \frac{(j+h)_k}{[j+h]_k},$$

where $[j+h]_k = [j+h]_q [j+h-1]_q \cdots [j+h-k+1]_q$.

By (2.5), we easily get

$$\begin{aligned}
(2.6) \qquad & q^x \beta_{n,q}^{(h,k)}(x \mid \lambda) \\
&= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_q^l \\
&\quad \times q^{x_1(h-1)+\cdots+x_k(h-k)+x} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} \Big\{ (q-1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_q^{l+1} \\
&\quad \times q^{x_1(h-2)+\cdots+x_k(h-k-1)} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&\quad + \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_q^l \, q^{x_1(h-2)+\cdots+x_k(h-1-k)} \\
&\quad \times d\mu_q(x_1)\cdots d\mu_q(x_k) \Big\} \\
&= \beta_{n,q}^{(h-1,k)}(x \mid \lambda) + \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}(q-1)\beta_{l+1,q}^{(h-1,k)}(x).
\end{aligned}$$

From (2.6), we have

$$(2.7) \qquad q^x \beta_{n,q}^{(h,k)}(x \mid \lambda) - \beta_{n,q}^{(h-1,k)}(x \mid \lambda) = (q-1) \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} \beta_{l+1,q}^{(h-1,k)}(x).$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$q^x \beta_{n,q}^{(h,k)}(x \mid \lambda) - \beta_{n,q}^{(h-1,k)}(x \mid \lambda) = (q-1)\sum_{l=0}^{n} S_1(n,l)\,\lambda^{n-l}\beta_{l+1,q}^{(h-1,k)}(x),$$

where $\beta_{n,q}^{(h,k)}(x)$ are Carlitz's higher-order q-Bernoulli polynomials.

We consider the polynomials $\beta_{n,q}^{(0,k)}(x \mid \lambda)$ in q^x

(2.8)

$$\begin{aligned}
&\beta_{n,q}^{(0,k)}(x \mid \lambda) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([x+x_1+\cdots+x_k]_q \mid \lambda\right)_n q^{-x_1-2x_2-\cdots-kx_k} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_q^l\, q^{-x_1-2x_2-\cdots-kx_k} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)(1-q)^{-l}\sum_{m=0}^{l}\binom{l}{m}(-1)^m q^{mx}\frac{m(m-1)\cdots(m-k+1)}{[m]_q[m-1]_q\cdots[m-k+1]_q} \\
&= \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l}{m}(-1)^m q^{mx}\lambda^{n-l}(1-q)^{k-l}S_1(n,l)\frac{(m)_k}{(q^m:q^{-1})_k}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\lim_{\lambda\to 0}\beta_{n,q}^{(0,k)}(x \mid \lambda) \\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m q^{mx}(1-q)^{k-n}\frac{(m)_k}{(q^m:q^{-1})_k} \\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m q^{mx}(1-q)^{-n}\frac{(m)_k}{[m]_k} \\
&= \beta_{m,q}^{(0,k)}(x).
\end{aligned}$$

By (2.8), we get

(2.9)

$$\begin{aligned}
&\beta_{n,q}^{(0,k)}(x+1 \mid \lambda) - \beta_{n,q}^{(0,k)}(x \mid \lambda) \\
&= \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l}{m}(-1)^m\lambda^{n-l}(1-q)^{-l}S_1(n,l)\left(q^{m(x+1)}-q^{mx}\right) \\
&\quad\times\frac{m(m-1)\cdots(m-k+1)}{[m]_q[m-1]_q\cdots[m-k+1]_q} \\
&= \sum_{l=1}^{n}\sum_{m=1}^{l}\binom{l}{m}(-1)^{m-1}\lambda^{n-l}(1-q)^{-l+1}S_1(n,l)\,q^{mx}[m]_q \\
&\quad\times\frac{m(m-1)\cdots(m-k+1)}{[m]_q[m-1]_q\cdots[m-k+1]_q}
\end{aligned}$$

$$= \sum_{l=0}^{n-1} \sum_{m=0}^{l} (-1)^m \lambda^{n-1-l} (1-q)^{-l} S_1(n, l+1) q^{(m+1)x} \binom{l}{m} (l+1) \frac{(m)_{k-1}}{[m]_{k-1}}$$
$$= \sum_{l=0}^{n-1} (l+1) q^x \left(\sum_{m=0}^{l} (-1)^m \lambda^{n-1-l} (1-q)^{-l} S_1(n, l+1) q^{mx} \binom{l}{m} \frac{(m)_{k-1}}{[m]_{k-1}} \right).$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$\beta_{n,q}^{(0,k)}(x+1 \mid \lambda) - \beta_{n,q}^{(0,k)}(x \mid \lambda)$$
$$= \sum_{l=0}^{n-1} (l+1) q^x \left(\sum_{m=0}^{l} (-1)^m \lambda^{n-1-l} (1-q)^{-l} S_1(n, l+1) q^{mx} \binom{l}{m} \frac{(m)_{k-1}}{[m]_{k-1}} \right).$$

Note that

$$\lim_{\lambda \to 0} \left(\beta_{n,q}^{(0,k)}(x+1 \mid \lambda) - \beta_{n,q}^{(0,k)}(x \mid \lambda) \right)$$
$$= nq^x \frac{1}{(1-q)^{n-1}} \sum_{m=0}^{n-1} (-1)^m q^{mx} \binom{n-1}{m} \frac{(m)_{k-1}}{[m]_{k-1}}$$
$$= nq^x \beta_{n-1,q}^{(0,k-1)}(x).$$

For $n \geq 0$, we have

$$\sum_{m=0}^{\infty} \beta_{m,q}^{(0,k)}(x \mid \lambda) \frac{t^m}{m!} \tag{2.10}$$
$$= \int_{\mathbb{Z}_p^k} (1+\lambda t)^{\frac{[x_1+\cdots+x_k+x]_q}{\lambda}} q^{-x_1-2x_2-\cdots-kx_k} d\mu_q(x_1) \cdots d\mu_q(x_k)$$
$$= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \lambda^{m-n} S_1(m,n) \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}^{(0,k)} \right) \frac{t^m}{m!}.$$

Thus, by (2.10), we get

$$\beta_{m,q}^{(0,k)}(x \mid \lambda) = \sum_{n=0}^{m} \sum_{l=0}^{n} \binom{n}{l} \lambda^{m-n} S_1(m,n) [x]_q^{n-l} q^{lx} \beta_{l,q}^{(0,k)}. \tag{2.11}$$

Note that

$$\lim_{\lambda \to 0} \beta_{m,q}^{(0,k)}(x \mid \lambda)$$
$$= \sum_{l=0}^{m} \binom{m}{l} [x]_q^{m-l} q^{lx} \beta_{l,q}^{(0,k)}$$
$$= \beta_{m,q}^{(0,k)}(x).$$

Let us consider the polynomials $\beta_{n,q}^{(h,1)}(x \mid \lambda)$ which are given by

$$\beta_{n,q}^{(h,1)}(x \mid \lambda) = \int_{\mathbb{Z}_p} \left([x+x_1]_q \mid \lambda \right)_n q^{x_1(h-1)} d\mu_q(x_1), \quad (n \geq 0). \tag{2.12}$$

From (2.12), we note that

$$\beta_{n,q}^{(h,1)}(x \mid \lambda) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x_1=0}^{p^N-1} \left([x+x_1]_q \mid \lambda\right)_n q^{x_1 h} \tag{2.13}$$
$$= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x_1=0}^{p^N-1} \lambda^n \left(\frac{[x_1+x]_q}{\lambda}\right)_n q^{x_1 h}$$
$$= \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x_1=0}^{p^N-1} [x+x_1]_q^l q^{x_1 h}$$
$$= \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) (1-q)^{-l} \sum_{m=0}^{l} \binom{l}{m} (-1)^m q^{mx} \frac{m+h}{[m+h]_q}$$
$$= \sum_{l=0}^{n} \sum_{m=0}^{l} \lambda^{n-l} S_1(n,l) (1-q)^{-l} \binom{l}{m} (-1)^m q^{mx} \frac{m+h}{[m+h]_q}.$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$\beta_{n,q}^{(h,1)}(x \mid \lambda) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} (-1)^m \lambda^{n-l} S_1(n,l) (1-q)^{-l} q^{mx} \frac{m+h}{[m+h]_q}.$$

Note that

$$\lim_{\lambda\to 0} \beta_{m,q}^{(h,1)}(x \mid \lambda) = \frac{1}{(1-q)^n} \sum_{m=0}^{n} \binom{n}{m} (-1)^m q^{mx} \frac{m+h}{[m+h]_q}$$
$$= \beta_{m,q}^{(h,1)}(x).$$

From (1.7), we have

(2.14)

$$q^h \beta_{n,q}^{(h,1)}(x+1 \mid \lambda) - \beta_{n,q}^{(h,1)}(x \mid \lambda)$$
$$= q^h \int_{\mathbb{Z}_p} \left([x+1+x_1]_q \mid \lambda\right)_n q^{x_1(h-1)} d\mu_q(x_1) - \int_{\mathbb{Z}_p} \left([x+x_1]_q \mid \lambda\right)_n q^{x_1(h-1)} d\mu_q(x_1)$$
$$= \lim_{N\to\infty} \frac{1}{[p^N]_q} \left(\sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) \left[p^N + x\right]_q^l q^{hp^N} - \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) [x]_q^l \right)$$
$$= (q-1) h \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) [x]_q^l + \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) \left(\left[p^N + x\right]_q^l - [x]_q^l \right)$$
$$= (q-1) h \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) [x]_q^l + \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l)\, l\, [x]_q^{l-1} q^x.$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$q^h \beta_{n,q}^{(h,1)}(x+1 \mid \lambda) - \beta_{n,q}^{(h,1)}(x \mid \lambda)$$

$$= (q-1)\, h \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l)\, [x]_q^l + \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l)\, l\, [x]_q^{l-1} q^x.$$

Note that

$$\lim_{\lambda\to 0} q^h \beta_{n,q}^{(h,1)}(x+1 \mid \lambda) - \beta_{n,q}^{(h,1)}(x \mid \lambda) = nq^x [x]_q^{n-1} + h(q-1)[x]_q^n.$$

Remark. The generating function of $\beta_{n,q}^{(h,1)}(x \mid \lambda)$ is given by

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+x_1]}{\lambda}} q^{x_1(h-1)} d\mu_q(x_1) = \sum_{n=0}^{\infty} \beta_{n,q}^{(h,1)}(x \mid \lambda) \frac{t^n}{n!}. \tag{2.15}$$

From (2.15), we note that

$$\beta_{n,q}^{(h,1)}(x \mid \lambda) = \int_{\mathbb{Z}_p} \left([x+x_1]_q \mid \lambda\right)_n q^{x_1(h-1)} d\mu_q(x_1), \tag{2.16}$$

where $n \geq 0$.

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